H2 Mathematics

GCE 'A' Levels

Mock Examination · Total: 98 marks · Generated 30/08/2026

Question 1

[11 marks]

CALCULUS (DIFFERENTIATION)

3 (Paper 2) The parametric equations of a curve are given by $x = a\sec\theta, y = a\tan\theta$, where a is a real constant.

(i) Show that $\frac{dy}{dx} = \operatorname{cosec}\theta$. [2]

(ii) Find the rate of change of y at $\theta = \frac{\pi}{6}$ when x is increasing at a constant rate of 2 units per second. [2]

(iii) Find the equation of the tangent to the curve at the point $R(a\sec\alpha,\ a\tan\alpha)$ and show that the equation of the normal to the curve is $y = 2a\tan\alpha - x\sin\alpha$. [2]

(iv) The tangent and normal at R meet the x-axis at T and N respectively. By finding the x-coordinates of T and N in the simplest form, show that $ON \cdot OT = 2a^2$. [5]

Question 2

[5 marks]

EQUATIONS AND INEQUALITIES

On a single diagram, sketch the graphs of $y = |2x - p|$ and $y = qx$ where the following conditions are satisfied, indicating the axial intercepts:

- p and q are constants, $p > 1$ and $q > 0$, and
- the graphs have only one point of intersection.

(a) State the least value of q.

(b) Solve the inequality $|2x - p| \le qx$, leaving your answer in terms of p and q.

Question 3

[10 marks]

INTEGRATION TECHNIQUES

(a) Find the exact value of $\int_0^1 \frac{x}{2x+2}\,dx$. [3]

(b) The expression $\frac{x^2}{(2-x)^2}$ can be written in the form $A + \frac{B}{2-x} + \frac{C}{(2-x)^2}$.

(i) Find the values of A, B and C. [3]

(ii) Show that $\int_0^1 \frac{x^2}{(2-x)^2}\,dx = p + q\ln 2$, where p and q are constants to be found. [4]

Question 4
[7 marks]

SAMPLING

(a) The supervisor of a petrol kiosk wants to monitor the service provided by his staff. He decides to choose a sample of 30 customers to take part in a survey at the payment counter before they leave the petrol kiosk.

(i) Describe how a systematic sample of 30 customers can be chosen from the first 120 customers in a day. [2]

(ii) Give a reason whether stratified sampling is appropriate in this situation. [1]

(b) The waiting time, in minutes, for a customer to be served at a petrol kiosk is a random variable with mean μ and standard deviation σ. The following are observed based on a random sample of 100 customers:

- for 95% of the time, the sample mean waiting time is between 7.5 minutes and 10 minutes;
- the probability that the sample mean waiting time is more than 10 minutes is two-thirds of the probability that the sample mean waiting time is less than 7.5 minutes.

Find the value of μ. [4]

Question 5
[11 marks]

VECTORS

A zip-line is a straight steel cable running from a platform at the point $A(0, 0, 40)$ to a platform at the point $B(60, 30, 10)$. Units are metres, the x- and y-axes are horizontal and the z-axis points vertically upwards. A vertical pole of height 40 m stands at the point $(20, 20, 0)$, so that the top of the pole is the point $T(20, 20, 40)$.

(a) Find a vector equation of the line AB and show that the length of the zip-line is $30\sqrt{6}$ m. [2]

(b) Find the coordinates of the point P on the zip-line which is nearest to T, and show that the shortest distance from T to the zip-line is $10\sqrt{2}$ m. [4]

(c) Find the acute angle between the zip-line and the horizontal. [2]

(d) Find a cartesian equation of the plane containing A, B and T. [3]

Question 6

[11 marks]

SEQUENCES AND SERIES

(a) A geometric progression has first term 9, and the tenth term is three times the fourth term. Find the 22nd term in the form $c\sqrt{3}$, where c is a non-zero constant to be determined. [3]

If all the terms of the geometric progression are positive and the kth term is more than 10^{40}, find the least value of k. [2]

(b) Runners A and B compete in a race by running laps around the school.

Runner A runs the first lap in 5 minutes 15 seconds but finds that his speed drops steadily so that each lap takes him d minutes more than the preceding one.

Runner B runs the first lap in 6 minutes and increases his speed steadily so that each successive lap takes him 0.99 of the time taken for the preceding one.

(i) Runner A takes 6 minutes 39 seconds to run the 8th lap. Find the value of d. [2]

(ii) Find expressions for the time taken for each runner to cover the first n laps. [2]

(iii) What is the least number of laps they complete if one of the runners took at least 38 minutes more than the other? [2]

Question 7

[7 marks]

DEFINITE INTEGRALS

(i) On the same axes, sketch the curves with equations $y = |2x^2 + 6x + 4|$ and $y = 3 - 4x$, indicating any intercepts with the axes and points of intersection. Hence solve the inequality $|2x^2 + 6x + 4| \geq 3 - 4x$.

(ii) Find the exact area bounded by the graphs of $y = 3 - 4x$, $y = |2x^2 + 6x + 4|$, $x = -3$ and $x = 1$.

Question 8

[10 marks]

FUNCTIONS

The functions f and g are defined by

$$f(x) = \frac{1}{2}(2^x) - 1 \text{ for } x \in \mathbb{R},\ x < 2, \qquad f(x) = 4 - \frac{3}{3x - 5} \text{ for } x \in \mathbb{R},\ x \geq 2,$$

$$g(x) : x \mapsto f(x) \text{ for } x \in \mathbb{R},\ x < a.$$

(i) Sketch the graph of $y = f(x)$, indicating clearly the axial intercepts and equations of asymptotes. [3]

(ii) State the maximum value of a such that g has an inverse. [1]

(iii) For the value of a found in (ii), find $g^{-1}(x)$ and state its domain. [3]

(iv) Given that the composite function f^2 exists, find f^2. [3]

[You do not need to simplify the expressions for f^2.]

Question 9

[10 marks]

MACLAURIN SERIES

In the triangle ABC, $AC = 1$, angle $BAC = x$ radians and angle $ACB = \frac{1}{6}\pi$ radians (see diagram).

(a) Show that $AB = \dfrac{1}{\cos x + \sqrt{3}\sin x}$. [3]

(b) Given that x is a sufficiently small angle, show that $AB \approx 1 + ax + bx^2$, for constants a and b to be determined exactly. [3]

(c) The first two terms in the Maclaurin series for $\ln(px + q)$ are equal to the first two terms in the series expansion in part (b). Using standard series from the List of Formulae (MF26), find the exact values of p and q. [4]

Question 10

[5 marks]

DISCRETE RANDOM VARIABLES

5 In 2011, a study suggested that approximately 92 per cent of the adult population are right-handed.

(i) Sixty samples, each consisting of 20 adults, are chosen from different parts of a cosmopolitan city. Find the probability that the total number of right-handed adults is less than 1100. [2]

(ii) Another sample consisting of n adults is taken, where $50 < n < 60$. Using a suitable approximation, find the largest value of n so that the probability that the number of right-handed adults is at most 48 is more than 0.4. [3]

Question 11

[11 marks]

CORRELATION AND LINEAR REGRESSION

10 Scientists are studying the growth of a particular species of bamboo. They want to understand how the height of the bamboo stalk changes over time. The table below shows the height, h inches, of a bamboo stalk at age, t weeks, after sprouting.

t	1	2	3	4	5	7	8	9
h	3.6	7.4	10.2	11.7	12.6	14.1	14.5	15.5

(a) Draw a scatter diagram of these data, and explain using the diagram why the relationship between h and t is not well-modelled by an equation of the form $h = a + bt$ or $h = a + bt^2$, where $b > 0$. [3]

(b) Consider the two models below:

Model A: $h = c + dt$,

Model B: $h = c + d\ln t$,

where c and d are constants. Explain which of Model A or Model B is a better fit to the data. State the equation of the least squares regression line in this case, giving your answer to 3 decimal places. [3]

(c) Use the equation of your regression line in (b) to estimate the length of a 6–week-old bamboo stalk. Comment on the reliability of this estimate. [2]

(d) Comment on the suitability of using this model in the long run. [1]

(e) Let H denote the height of the bamboo stalk calculated for each value of t in the table using the least squares regression line obtained in (b). Find the value of S, where $S = \sum(h - H)^2$, giving your answer to 2 decimal places. [1]

(f) For each of the eight sample values of t, $H^{'}$ is given by $H^{'} = p + q\ln t$, where p and q are constants. Would $\sum(h - H^{'})^2$ be greater or less than S and why? [1]

Answers

Question 1

(i) $\dfrac{dx}{d\theta} = a\sec\theta\tan\theta$ and $\dfrac{dy}{d\theta} = a\sec^2\theta$, so

$$\frac{dy}{dx} = \frac{a\sec^2\theta}{a\sec\theta\tan\theta} = \frac{\sec\theta}{\tan\theta} = \frac{1}{\sin\theta} = \operatorname{cosec}\theta$$

(ii) $\dfrac{dy}{dt} = \dfrac{dy}{dx}\cdot\dfrac{dx}{dt} = \operatorname{cosec}\dfrac{\pi}{6}\times 2 = 2\times 2 = 4$ units per second.

(iii) The tangent at R has gradient $\operatorname{cosec}\alpha$:

$$y - a\tan\alpha = \operatorname{cosec}\alpha\,(x - a\sec\alpha), \quad \text{i.e.} \quad y = x\operatorname{cosec}\alpha + a\tan\alpha - a\sec\alpha\operatorname{cosec}\alpha$$

The normal has gradient $-\sin\alpha$:

$$y - a\tan\alpha = -\sin\alpha\,(x - a\sec\alpha) = -x\sin\alpha + a\tan\alpha$$

since $a\sin\alpha\sec\alpha = a\tan\alpha$. Hence $y = 2a\tan\alpha - x\sin\alpha$.

(iv) For N, put $y = 0$ in the normal: $x = \dfrac{2a\tan\alpha}{\sin\alpha} = 2a\sec\alpha$, so $ON = 2a\sec\alpha$.

For T, put $y = 0$ in the tangent:

$$x = a\sec\alpha - a\tan\alpha\sin\alpha = \frac{a}{\cos\alpha}\left(1 - \sin^2\alpha\right) = a\cos\alpha$$

so $OT = a\cos\alpha$. Therefore $ON\cdot OT = 2a\sec\alpha\times a\cos\alpha = 2a^2$.

Question 2

(a) For the graphs to have only one point of intersection, the line $y = qx$ must have gradient at least that of $y = 2x - p$, i.e. $q \geq 2$. The least value of q is 2.

(b) For $x < \frac{p}{2}$, $|2x - p| = p - 2x$. At the intersection, $p - 2x = qx$, giving $x = \frac{p}{q+2}$. Hence $|2x - p| \leq qx$ for $x \geq \frac{p}{q+2}$.

Question 3

(a) $\dfrac{x}{2x+2} = \dfrac{1}{2} - \dfrac{1}{2x+2}$, so

$$\int_0^1 \frac{x}{2x+2}\,dx = \left[\frac{1}{2}x - \frac{1}{2}\ln(2x+2)\right]_0^1 = \left(\frac{1}{2} - \frac{1}{2}\ln 4\right) - \left(-\frac{1}{2}\ln 2\right) = \frac{1}{2} - \frac{1}{2}\ln 2.$$

(b)(i) $x^2 = A(2-x)^2 + B(2-x) + C$. Setting $x = 2$ gives $C = 4$. Comparing x^2 terms gives $A = 1$; comparing x terms, $-4A - B = 0 \Rightarrow B = -4$. Hence $A = 1, B = -4, C = 4$.

(b)(ii)

$$\int_0^1 \left(1 - \frac{4}{2-x} + \frac{4}{(2-x)^2}\right) dx = \left[x + 4\ln(2-x) + \frac{4}{2-x}\right]_0^1$$

$$= (1 + 0 + 4) - (0 + 4\ln 2 + 2) = 3 - 4\ln 2.$$

Hence $p = 3$ and $q = -4$.

Question 4

(a)(i) Since $\dfrac{120}{30} = 4$, choose the first customer at random from the first 4 customers (by simple random sampling), then select every 4th customer thereafter until 30 customers have been chosen.

(a)(ii) No. The profiles of the customers cannot be gathered and broken down into strata proportional to that of the 120 customers, since the survey is carried out just before the customers leave.

(b) Let $\bar{X}$ be the sample mean waiting time. Since $n = 100$ is large, by the Central Limit Theorem $\bar{X} \sim N\left(\mu, \dfrac{\sigma^2}{100}\right)$ approximately.

Let $P(\bar{X} < 7.5) = x$. Then $P(\bar{X} > 10) = \dfrac{2}{3}x$, and $x + 0.95 + \dfrac{2}{3}x = 1$, giving $x = 0.03$. So $P(\bar{X} < 7.5) = 0.03$ and $P(\bar{X} > 10) = 0.02$.

Standardising with $\bar{X}$ having standard deviation $\dfrac{\sigma}{10}$:

$$\frac{7.5 - \mu}{\sigma/10} = -1.8808, \qquad \frac{10 - \mu}{\sigma/10} = 2.0537.$$

That is $10\mu = 1.8808\sigma + 75$ and $10\mu = 100 - 2.0537\sigma$. Equating gives $3.9345\sigma = 25$, so $\sigma = 6.35$. Substituting back, $\mu = 8.70$ (3 s.f.).

Question 5

(a) $\overrightarrow{AB} = \begin{pmatrix} 60 \\ 30 \\ -30 \end{pmatrix} = 30 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, so

$$= \begin{pmatrix} 0 \\ 0 \\ 40 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \qquad t \in \mathbb{R}.$$

$\left|\overrightarrow{AB}\right| = 30\sqrt{2^2 + 1^2 + (-1)^2} = 30\sqrt{6}$ m (shown).

(b) A general point of AB is $P(2t,\ t,\ 40 - t)$, so $\overrightarrow{TP} = \begin{pmatrix} 2t - 20 \\ t - 20 \\ -t \end{pmatrix}$.

$\overrightarrow{TP} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 4t - 40 + t - 20 + t = 6t - 60 = 0 \Rightarrow t = 10$, which satisfies $0 \leqslant t \leqslant 30$, so P lies on the cable.

$P(20, 10, 30)$ and $\overrightarrow{TP} = \begin{pmatrix} 0 \\ -10 \\ -10 \end{pmatrix}$, giving $TP = \sqrt{200} = 10\sqrt{2}$ m (shown).

(c) With $= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ normal to the horizontal plane, $\sin\alpha = \dfrac{|-1|}{\sqrt{6}}$, so $\alpha = 24.1^\circ$ (1 d.p.).

(d) $\overrightarrow{AT} = \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix}$ and a normal is $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} = \begin{pmatrix} 20 \\ -20 \\ 20 \end{pmatrix} \parallel \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Using $A(0, 0, 40)$: $0 - 0 + 40 = 40$, so the plane is $x - y + z = 40$ (checks: B gives $60 - 30 + 10 = 40$, T gives $20 - 20 + 40 = 40$).

Question 6

(a) With $a = 9$ and common ratio r: $9r^9 = 3\left(9r^3\right) \Rightarrow r^6 = 3 \Rightarrow r = \pm 3^{1/6}$.

$$u_{22} = 9r^{21} = 9\left(r^6\right)^3 r^3 = 243r^3 = \pm 243\sqrt{3},$$

so $c = 243$ (or $c = -243$ if r is negative).

For all terms positive, $r = 3^{1/6}$ and $u_k = 9 \cdot 3^{(k-1)/6} = 3^{(k+11)/6}$. Then

$$\frac{k+11}{6}\lg 3 > 40 \Rightarrow k + 11 > 503.02 \Rightarrow k > 492.02,$$

so the least value is $k = 493$.

(b)(i) Runner A is arithmetic with first term 5.25 min. The 8th lap takes $a + 7d = 6.65 \Rightarrow 7d = 1.4 \Rightarrow d = 0.2$.

(ii) Runner A: $S_A = \frac{n}{2}[2(5.25) + (n-1)(0.2)] = 0.1n^2 + 5.15n$ minutes.

Runner B: $S_B = \dfrac{6\left(1 - 0.99^n\right)}{1 - 0.99} = 600\left(1 - 0.99^n\right)$ minutes.

(iii) Require $|S_A - S_B| \geq 38$. At $n = 20$: $143 - 109.25 = 33.75 < 38$. At $n = 21$: $152.25 - 114.16 = 38.1 \geq 38$. Least number of laps $= 21$.

Question 7

(i) The line $y = 3 - 4x$ meets $y = |2x^2 + 6x + 4|$ where $3 - 4x = 2x^2 + 6x + 4$, i.e. $2x^2 + 10x + 1 = 0$, giving $x = -0.102$ and $x = -4.90$ (3 s.f.); the equation $3 - 4x = -(2x^2 + 6x + 4)$ has no real roots. The intersection points are approximately $(-4.90, 22.6)$ and $(-0.102, 3.41)$.

Solution of $|2x^2 + 6x + 4| \geq 3 - 4x$: $x \leq -4.90$ or $x \geq -0.102$.

(ii) Splitting the region and integrating the difference between the line and $|2x^2 + 6x + 4|$ over $[-3, 1]$ gives the exact area

$$\text{Area} = \frac{20}{3}\text{ units}^2.$$

Question 8

(i) For $x < 2, y = 2^{x-1} - 1$ is increasing with x-intercept $(1, 0)$, y-intercept $\left(0, -\frac{1}{2}\right)$ and horizontal asymptote $y = -1$ as $x \to -\infty$; it approaches the value 1 as $x \to 2^-$.

For $x \geq 2, y = 4 - \frac{3}{3x-5}$ is increasing from the endpoint $(2, 1)$ with horizontal asymptote $y = 4$.

(ii) Maximum value of a is 1.

(iii) For $x < 1, g(x) = \frac{1}{2}\left(2^x\right) - 1$. Let $y = \frac{1}{2}\left(2^x\right) - 1$, so

$$2^x = 2(1+y) \;\Rightarrow\; x = \log_2\left[2(1+y)\right] = 1 + \log_2(1+y).$$

Hence

$$g^{-1}(x) = 1 + \log_2(1+x), \qquad D_{g^{-1}} = R_g = (-1, 0).$$

(iv) Solving $f(x) = 2$ on the second branch: $4 - \frac{3}{3x-5} = 2$ gives $x = \frac{13}{6}$. Hence

for $x < 2$: $f^2(x) = \frac{1}{2}\left(2^{\frac{1}{2}(2^x)-1}\right) - 1 = 2^{\,2^{x-1}-2} - 1$;

for $2 \leq x < \frac{13}{6}$: $f^2(x) = \frac{1}{2}\left(2^{\,4-\frac{3}{3x-5}}\right) - 1 = 2^{\,3-\frac{3}{3x-5}} - 1$;

for $x \geq \frac{13}{6}$: $f^2(x) = 4 - \frac{3}{3\left(4-\frac{3}{3x-5}\right)-5} = 4 - \frac{9x-15}{21x-44}$.

Question 9

(a) Angle $ABC = \pi - x - \dfrac{\pi}{6} = \dfrac{5\pi}{6} - x$. By the sine rule, $\dfrac{AB}{\sin\frac{\pi}{6}} = \dfrac{AC}{\sin\left(\frac{5\pi}{6} - x\right)}$.

$$AB = \frac{\sin\frac{\pi}{6}}{\sin\left(\frac{5\pi}{6} - x\right)} = \frac{\frac{1}{2}}{\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x} = \frac{1}{\cos x + \sqrt{3}\sin x}. \text{ (shown)}$$

(b) For small x, $\cos x \approx 1 - \dfrac{1}{2}x^2$ and $\sin x \approx x$, so

$$AB \approx \;1 + \;\sqrt{3}x - \frac{1}{2}x^2 \quad ^{-1} \approx 1 - \;\sqrt{3}x - \frac{1}{2}x^2 \; + \left(\sqrt{3}x\right)^2 = 1 - \sqrt{3}x + \frac{7}{2}x^2.$$

Hence $a = -\sqrt{3}$ and $b = \dfrac{7}{2}$.

(c) $\ln(px+q) = \ln\; q\;\; 1 + \dfrac{p}{q}x \;\; = \ln q + \ln\;\; 1 + \dfrac{p}{q}x \;\; \approx \ln q + \dfrac{p}{q}x$.

Comparing with $1 - \sqrt{3}x$: $\ln q = 1$ and $\dfrac{p}{q} = -\sqrt{3}$. Hence $q = e$ and $p = -\sqrt{3}e$.

Question 10

(i) Let X be the number of right-handed adults in a sample of 20, so $X \sim B(20, 0.92)$ with $E(X) = 18.4$ and $Var(X) = 20(0.92)(0.08) = 1.472$.

Let $T = X_1 + X_2 + \quad + X_{60}$. Since $n = 60$ is large, by the Central Limit Theorem

$$T \sim N(18.4 \times 60,\ 1.472 \times 60) = N(1104,\ 88.32) \text{ approximately.}$$

$$P(T < 1100) = 0.335.$$

(ii) Let Y be the number of adults who are not right-handed out of n, so $Y \sim B(n, 0.08)$. Since n is large and $0.08n < 4.8 < 5$ for $50 < n < 60$, $Y \sim Po(0.08n)$ approximately.

The number of right-handed adults is $n - Y$, so the requirement is

$$P(n - Y \le 48) > 0.4 \Leftrightarrow P(Y \ge n - 48) > 0.4 \Leftrightarrow P(Y \le n - 49) < 0.6.$$

From the GC: $n = 52$ gives $P(Y \le 3) = 0.40285$; $n = 53$ gives $P(Y \le 4) = 0.58206$; $n = 54$ gives $P(Y \le 5) = 0.73333$.

Hence the largest value is $n = 53$.

Question 11

(a) The scatter diagram of h against t shows points rising steeply from $(1, 3.6)$ and then flattening towards $(9, 15.5)$, with both axes labelled and scaled.

As t increases, h increases at a decreasing rate, so the points lie on a curve rather than a straight line and $h = a + bt$ is not suitable. For $h = a + bt^2$ with $b > 0$, h would increase at an increasing rate, which is the opposite of the observed trend, so that model is also unsuitable.

(b) For Model A, $r = 0.974$; for Model B, $r = 0.997$ (3 s.f.). The product moment correlation coefficient for Model B is closer to 1, so Model B is the better fit. The least squares regression line of h on $\ln t$ is

$$h = 3.890 + 5.311 \ln t \text{ (3 d.p.).}$$

(c) When $t = 6$, $h = 3.890 + 5.311 \ln 6 = 13.4$ inches (3 s.f.). The estimate is reliable because $t = 6$ lies within the data range $1 \le t \le 9$, so this is interpolation, and $r = 0.997$ is very close to 1, indicating a strong linear relationship between h and $\ln t$.

(d) The model is not suitable in the long run: as $t \to \infty$, $\ln t \to \infty$ and so $h \to \infty$, but the height of a bamboo stalk cannot increase indefinitely.

(e) Evaluating $H = 3.890166 + 5.311381 \ln t$ at each of the eight values of t and summing the squared residuals gives

$$S = \sum (h - H)^2 = 0.77 \text{ (2 d.p.).}$$

(f) Greater. The line found in (b) is by definition the line of the form $p + q \ln t$ that minimises the sum of squared vertical deviations, so for any other constants p and q the value of $\sum (h - H')^2$ cannot be smaller than S; it is greater unless p and q are exactly the least squares values.